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Consistent Positive and Linear Positive Quasi-antiorders

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Abstract

In this paper we introduce notions and some basic properties of consistent positive and linear positive quasi-antiorders on semigroups with apartness.

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1 Introduction and Preliminaries

This investigation, in Bishop's constructive mathematics in sense of well-known books [2], [3], [6] and Romano's papers [9]-[15], is continuation of forthcoming Crvenkovic, Mitrovic and Romano's paper [4], and the Romano's paper [16]. Bishop's constructive mathematics is developed on Constructive logic (or Intuitionistic logic ([19])) - logic without the Law of Excluded Middle $P \vee \neg\neg P$. Let us note that in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic.

A relation q on S is a *coequality relation* on S if and only if it is consistent, symmetric and cotransitive ([9]). Let $(S, =, \neq, \cdot)$ be a semigroup with an apartness. A relation τ on S is a *quasi-antiorder* ([9], [13]-[15]) on S if

$$\tau \subseteq \neq, \;\; \tau \subseteq \tau * \tau,$$

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where the operation "$*$" is *filled product* of relations (For more information about the filled product and the notion of cotransitive internal fulfilment of a relation, the reader can find in any of the following papers [10]-[13].) If τ is a quasi-antiorder on S compatible with the semigroup operation in S, then ([14]) the relation $q = \tau \cup \tau^{-1}$ is an anticongruence on S. Firstly, the relation $q^C = \{(x, y) \in S \times S : (x, y) \bowtie q\}$ is a congruence on S compatible with q, in the following sense $q^C \circ q \subseteq q$ and $q \circ q^C \subseteq q$. It is easy to establish the isomorphism $S/(q^C, q) \cong S/q$ between semigroups.

A subset A of a semigroup S is *consistent* if for $a, b \in S$, $ab \in A$ implies $a \in A \wedge b \in A$. It is easy to check if A is a consistent subset of S, then A^C is a subsemigroup of S. Opposite assertion, "If T is a subsemigroup of S, then T^C is a consistent subset of S" does not hold in general. A consistent subsemigroup F of S will be called a *filter* of S. In that case, the subset F^C is a completely prime ideal of S. The opposite assertion "if J is a completely prime ideal of S, then J^C is a filter of S" does not hold in general.

In the Classical Semigroup Theory concept of positive quasi-order has been introduced by B. M. Schein. After that, positive quasi-orders have been studied from different points of view by many authors, mainly by T. Tamura [17], [18], M. S. Putcha [8], and S.Bodganovic and M.Ciric [5]. Quasi-antiorder relation in semigroup with apartness the first time was defined by this author in his paper [9]. Further on, the investigation on basic properties of quasi-antiorder relations on sets and semigroups with apartness is done by the author in his papers [13]-[15] and in forthcoming paper [16]. Positive quasi-antiorder is studied by Crvenkovic, Mitrovic and Romano in their forthcoming paper [4].

For undefined notion and notations we referee to books [1], [2], [3], [7], and [19] and to papers [9]-[15].

In this paper we study a constructive aspect of positive quasi-antiorders of semigroups with apartness. The notions of consistent positive and linear positive quasi-antiorders and some basic properties of them are shown.

2 Positive quasi-antiorders

By a quasi-antiorder we mean a consistent cotransitive relation on a set. For a quasi-antiorder τ on a semigroup S we say that it is compatible with the semigroup operation if and only if

$$(\forall a, b, x \in S)(((ax, bx) \in \tau \Longrightarrow (a, b) \in \tau) \wedge ((xa, xb) \in \tau \Longrightarrow (a, b) \in \tau)).$$

A relation τ on a semigroup S is called *positive* ([4]) if and only if

$$(\forall a, b \in S)((a, ab) \bowtie \tau \ \wedge \ (b, ab) \bowtie \tau)),$$

and it is called *lower-potent* ([15]) if

$$(a^n, a) \bowtie \tau,$$

for any $a \in S$ and any $n \in N$. In the paper [15], it is shown that if τ is a quasi-antiorder compatible with the semigroup operation, then it is lower-potent if and only if $(a^2, a) \bowtie \tau$, for all $a \in S$.

Theorem 2.0 ([4], Theorem 3.4) *The following conditions for a quasi-antiorder τ on a semigroup S are equivalent:*
(1) is positive;
(2) $(\forall a, b \in S)(a\tau \cup b\tau \subseteq (ab)\tau)$*;*
(3) $(\forall a, b \in S)(\tau(ab) \subseteq \tau a \cap \tau b)$*;*
(4) $a\tau$ is a strongly extensional consistent subset of S such that $a \bowtie a\tau$ for each $a \in S$; and
(5) τb is a strongly extensional ideal of S such that $b \bowtie \tau b$, for each $b \in S$.

In the following we will describe a construction of maximal positive quasi-antiorder relation on a semigroup S. Let a and b be elements of S. Then ([12], Theorem 6) the set $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S. The consistent subset $C_{(a)}$ is called a *principal consistent subset* of S generated by a. We introduce relation f defined by $(a, b) \in f \Longleftrightarrow b \in C_{(a)}$. For an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A_n(a) = \{xS : (a, x) \in {}^n f\}, \ A(a) = \{x \in S : (a, x) \in c(f)\}$$

$$B_n(a) = \{y \in S : (y, a) \in {}^n f\}, \quad B(a) = \{y \in S : (y, a) \in c(f)\}.$$

In the following two lemmas we will present some basic characteristics of these sets.

Theorem 2.1 ([12], [4]) *Let a and b be elements of a semigroup S. Then:*
(1) Then the set $A(a) = \bigcap_{n \in N} A_n(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie A(a)$.
(2) $A(a) \cup A(b) \subseteq A(ab)$*.*
(3) Then the set $B(a) = \bigcap_{n \in N} B_n(a)$ is the maximal strongly extensional ideal of S such that $a \bowtie B(a)$.
(4) $B(ab) \subseteq B(a) \cap B(b)$*.*

(5) The relation $c(f)$ is the maximal positive quasi-antiorder relation on semigroup S.
(6) ([4], Theorem 2.4) A quasi-antiorder τ on a semigroup S is positive if and only if it contained in the maximal quasi-antiorder relation $c(f)$ on S.

Besides, we will describe maximal lower-potent positive quasi-antiorder relation on a semigroup S. Let S be a semigroup with apartness and let a, b be arbitrary elements of S. As mentioned above in this section, the set $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S called a *principal* consistent subset of S generated by the element a, and $cr(C_{(a)}) = \{x \in C(a) : (\forall n \in N)(x^n \in C_{(a)})\}$ called a *coradical* of principal consistent subset of S generated by the element a. We introduce relation s defined by $(a,b) \in s \Longleftrightarrow b \in cr(C_{(a)})$, and we will describe some properties of relation $c(s)$.

For an element a of a semigroup S and for $n \in N$ we introduce the following notations:

$$D_n(a) = \{x \in S : (a,x) \in {}^n s\},\ D(a) = \{x \in S : (a,x) \in c(s)\}$$

$$E_n(a) = \{x \in S : (x,a) \in {}^n s\}, E(a) = \{x \in S : (x,a) \in c(s)\}$$

By the following results we will present some basic characteristics of these sets.

Theorem 2.2 ([12]) *The relation $c(s)$ satisfies the following properties:*
(1) $c(s)$ is a consistent relation on S.
(2) $c(s)$ is a cotransitive relation.
(3) $(\forall n \in N)((a,a^n) \bowtie c(s))$.
(4) Then the set $D(a) = \bigcap_{n \in N} D_n(a)$ is the maximal strongly extensional consistent potent semifilter of S such that $a \bowtie D(a)$.
(5) $D(a) \cup D(b) \subseteq D(ab)$.
(6) $(\forall n \in N)(D(a) = D(a^n))$.
(7) Then the set $E(a) = \bigcap_{n \in N} E_n(a)$ is the maximal strongly extensional completely potent semiprime ideal of S such that $a \bowtie E(a)$.
(8) $E(ab) \subseteq E(a) \cap E(b)$.
(9) $(\forall n \in N)(E(a^n) = E(a))$.
(10) The relation $c(s)$ is the maximal positive quasi-antiorder relation on semigroup S and the following $(\forall a \in S)(\forall n \in N)((a^n,a) \bowtie c(s))$ holds.
(11) A positive quasi-antiorder τ on a semigroup S is lower-potent positive if and only if it contained in the maximal lower-potent positive quasi-antiorder relation $c(s)$ on S.

Using Tamura's idea from [18], in paper [15], for given relation σ in semigroup S with apartness we construct the following relation

$$(a,b) \in p(\sigma) \iff (\forall n \in N)((a,b^n) \in \sigma).$$

A sense of such a method is to build a lower-potent quasi-antiorder from a given relation σ. Namely, the following assertion can be easily proved:

Theorem 2.3 (1) ([15], Theorem 3.1) *The maximal lower-potent quasi-antiorder on a semigroup S contained in σ on S equals $cp(\sigma \cap \neq)$.*
(2) ([15], Theorem 3.2) *The maximal lower-potent positive quasi-antiorder on a semigroup S contained in a relation σ on S equals $cp(\sigma \cap f)$.*

3 Consistent positive and linear positive quasi-antiorders

The following proposition gives some equalities about positive quasi-antiorder:

Theorem 3.1 *Let τ is positive quasi-antiorder in semigroup S. Then the following conditions are equivalent:*
(1) $(\forall a,b \in S)((ab)\tau = a\tau \cup b\tau)$;
(2) τb is strongly extensional completely prime ideal of S for every b in S;
(3) $(\forall a,b,c \in S)((ab,c) \in \tau \Longrightarrow ((a,c) \in \tau \vee (b,c) \in \tau))$.
Proof.
(1) $\Longrightarrow$ (2) Let $xy \in \tau b$. Then, $b \in (xy)\tau = x\tau \cup y\tau$. Thus, $b \in x\tau$ or $b \in y\tau$. So, $x \in \tau b$ or $y \in \tau b$.
(2) $\Longrightarrow$ (1) If x is an arbitrary element of $(ab)\tau$, then $ab \in x$ and $a \in x\tau \vee b \in x\tau$ because $x\tau$ is a strongly extensional completely prime ideal of S. Therefore, the following implication $x \in (ab)\tau \Longrightarrow x \in a\tau \cup b\tau$ holds.
(3) $\Longrightarrow$ (1) Let (3) holds. Then, for $x \in (ab)\tau$ we have $(ab,x) \in \tau$. Thus, $(a,x) \in \tau$ or $(b,x) \in \tau$. Hence, finally, we have $x \in a\tau \cup b\tau$.
(1) $\Longrightarrow$ (3) Let the formula (1) is valid. Suppose that a, b and c are elements of S such that $(ab,c) \in \tau$. Then, $c \in (ab)\tau = a\tau \cup b\tau$. So, we have $c \in a\tau \vee c \in b\tau$ and, finally $(a,c) \in \tau$ or $(b,c) \in \tau$. $\square$

Corollary 3.1.1 *If τ is a quasi-antiorder relation on semigroup S which satisfies one of conditions (1), (2) or (3) in the above theorem, then the following implication holds:*
(4)$(\forall a,b,c \in S)((a,c) \bowtie \tau \wedge (b,c) \bowtie \tau \Longrightarrow (ab,c) \bowtie \tau)$.
Proof: Let a, b and c be elements of S such that $(a,c) \bowtie \tau$ and $(b,c) \bowtie \tau$,

and let (u, v) be an arbitrary element of τ. Then:
$(u, v) \Longrightarrow (u, ab) \in \tau \vee (ab, c) \in \tau \vee (c, v) \in \tau$
$\Longrightarrow u \neq ab \vee (a, c) \in \tau \vee (b, c) \in \tau \vee c \neq v$
$\Longrightarrow (ab, c) \neq (u, v) \in \tau.\square$

Remark I: In the Classical semigroup theory, if positive quasi-order σ satisfies the following condition

$$(\forall a, b, c \in S)((a, c) \in \sigma \wedge (b, c) \in \sigma \Longrightarrow (ab, c) \in \sigma),$$

then for σ it is called that it satisfies the *cm-property* (common multiple property). So, we can define a new kind of positive quasi-antiorder relation: positive quasi-antiorder with cm-property. But, there is a problem with constructive definition of the new notion: If we choose the formula (4) as the definition of new kind of positive quasi-antiorder, then we lose results of the Theorem 3.1. If we choose the formula (3) in the Theorem 3.1 as the definition of new kind of positive quasi-antiorder, results of Theorem 3.1 are preserved, but this definition is stronger (more demanding) then in the Classical case. Besides, in the second case, the notion "common multiple" is not adequate notion for a positive quasi-antiorder satisfying the formula (3). It would be better to using the following notion *consistent positive quasi-antiorder*, in my opinion.

In the following theorem we give a description of another special quasi-antiorder relation:

Theorem 3.2 *Let τ be a positive quasi-antiorder in a semigroup S. Then, the following conditions are equivalent:*
(a) $(\forall a, b \in S)((ab, a) \bowtie \tau \vee (ab, b) \bowtie \tau)$;
(b) $(\forall a, b \in S)(\tau(ab) = \tau a \cap \tau b)$;
(c) $a\tau$ *is a strongly extensional filter of* S *for every* a *in* S;
(d) $(\forall a, b \in S)((a, b) \bowtie \tau \vee (b, a) \bowtie \tau)$.
Proof:
$(a) \Longrightarrow (b)$
$x \in \tau a \cap \tau b \Longleftrightarrow (x, a) \in \tau \wedge (x, b) \in \tau$
$\Longrightarrow ((x, ab) \in \tau \vee (ab, a) \in \tau) \wedge ((x, ab) \in \tau \vee (ab, b) \in \tau)$
$\Longrightarrow (x, ab) \in \tau$
$\Longleftrightarrow x \in \tau(ab)$.
$(b) \Longleftrightarrow (c)$
$x \in a\tau \wedge y \in a\tau \Longleftrightarrow a \in \tau x \wedge a \in \tau y$
$\Longleftrightarrow a \in \tau x \cap \tau y = \tau(xy)$
$\Longleftrightarrow xy \in a\tau$.

$(c) \Longrightarrow (a)$ Let (u,v) be an arbitrary element of τ and let a, b, c be arbitrary elements of S. Then, $((u,ab) \in \tau \vee (ab,a) \in \tau \vee (a,v) \in \tau)$ and $((u,ab) \in \tau \vee (ab,b) \in \tau \vee (b,v) \in \tau)$, and thus,

$$(u,v) \neq (ab,a) \vee (u,v) \neq (ab,b) \vee (ab \in \tau a \cap \tau b = \tau(ab)).$$

So, we have $(ab,a) \bowtie \tau$ or $(ab,b) \bowtie \tau$, since $ab \bowtie \tau(ab)$.
$(a) \Longleftrightarrow (d)$ Out of (d) immediately follows (a). Let (d) holds for elements a, b of semigroup S and let (u,v) be an arbitray element of τ. Particularly, we have $(a^2,a) \bowtie \tau$ and $(a,a^2) \bowtie \tau$. Thus,

$$((u,ab) \in \tau \vee (ab,aa) \in \tau \vee (aa,a) \in \tau \vee (a,v) \in \tau) and$$

$$((u,ab) \in \tau \vee (ab,bb) \in \tau \vee (bb,b) \in \tau \vee (b,v) \in \tau).$$

Hence,

$$(u \neq ab \vee (b,a) \in \tau, \vee a \neq v) \wedge (u \neq ab \vee (a,b) \in \tau \vee b \neq v).$$

So, we have $(ab,a) \neq (u,v)$ or $(ab,b) \neq (u,v)$.□

Remark II: In the Classical semigroup theory, for quasi-order σ we say that it is linear if and only the following formula holds:

$$(\forall a,b \in S)((a,b) \in \sigma \vee (b,a) \in \sigma).$$

So, we have the possibility for establish a new kind of positive quasi-antiorder relation on semigroup with apartness: For quasi-antiorder relation τ on semigroup S we say that it is a *linear* if and only the following formula

$$(\forall a,b \in S)((a,b) \bowtie \tau \vee (b,a) \bowtie \tau)$$

holds. Therefore, for linear positive quasi-antiorder on semigroup S we have equivalent descriptions (a), (b) and (c).

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